Hepp and Speer sectors within modern strategies of sector decomposition

JHEP05(2009)004

(http://iopscience.iop.org/1126-6708/2009/05/004)

Published by IOP Publishing for SISSA

Received: *January 21, 2009*
Revised: *March 27, 2009*
Accepted: *April 14, 2009*
Published: *May 5, 2009*

Hepp and Speer sectors within modern strategies of sector decomposition

A.V. Smirnov[a] **and V.A. Smirnov**[b]

[a] *Scientific Research Computing Center, Moscow State University, Leninskie gory, Moscow, Russia*

[b] *Nuclear Physics Institute of Moscow State University, Moscow State University, Leninskie gory, Moscow, Russia*

E-mail: asmirnov80@gmail.com, smirnov@theory.sinp.msu.ru

Abstract: Hepp and Speer sectors were successfully used in the sixties and seventies for proving mathematical theorems on analytically or/and dimensionally regularized and renormalized Feynman integrals at Euclidean external momenta. We describe them within recently developed strategies of introducing iterative sector decompositions. We prove that Speer sectors are reproduced within one of the existing strategies.

Keywords: NLO Computations, QCD

ArXiv ePrint: 0812.4700

doi:10.1088/1126-6708/2009/05/004

Contents

1 Introduction

The so-called alpha representation [1–11] was initially used to introduce dimensional regularization [12, 13] and to prove various mathematical results on Feynman integrals [3–6, 8–11]. The standard way to analyze convergence of Feynman integrals is to decompose the initial integration domain of alpha parameters into appropriate subdomains (sectors) and introduce, in each sector, new variables in such a way that the integrand factorizes, i.e. becomes equal to a monomial in new variables times a non-singular function. This procedure turned out to be successful for Euclidean external momenta, i.e with $(\sum q_i)^2 < 0$ for any partial sum, when the sectors of Hepp [3] and Speer [5] were introduced.

However, in practice, one often deals with Feynman integrals on a mass shell or at a threshold. In this case, Hepp or Speer sectors generally do not provide a factorization of the integrand so that the analysis of convergence fails within this technique. Therefore, general theorems on such 'physical' Feynman integrals could not be proved up to now.

Recently Binoth and Heinrich introduced sector decompositions of a new kind [14]. (See [15] for a review.) They provided a powerful method of evaluating Feynman integrals numerically in situations with severe UV, IR and collinear divergences. In contrast to Hepp and Speer sectors, the sectors of [14] are introduced iteratively, according to so-called *sector decomposition strategies*. The corresponding algorithm was implemented on a computer. Although this algorithm was successfully applied to numerically evaluate complicated Feynman integrals and to check existing analytical results (see, e.g. [16–18]) it was unclear where this iterative procedure stops at some point, i.e. results in the factorization of the integrands so that one can apply it for numerical evaluation. Indeed, in some examples, closed loops appear within this algorithm.

The first algorithm guaranteed to terminate was developed by Bogner and Weinzierl [19]. More precisely, certain strategies within this algorithm guarantee that closed loops do not appear. The algorithm works at least if squares $(\sum q_i)^2$ of partial sums of the external momenta are either negative or zero.[1] (We will formulate a more general condition in the next section.) The corresponding computer code is public. The second public algorithm [20] called `FIESTA` provided one more strategy (Strategy S) for sector decompositions which also leads to a factorization of the integrand for general Feynman integrals. It was successfully applied in [21].

The purpose of this paper is to describe Hepp and Speer sectors in an iterative way, within the modern technique of sector decompositions. In the next section, we describe the alpha representation and introduce the corresponding graph-theoretical notation. Then, in section 3, we establish the connection of Hepp and Speer sectors with iterative sector decomposition. In conclusion, we discuss some perspectives. In appendix, we prove a theorem stating that Speer sectors are reproduced within Strategy S.

2 Parametric representations and graph-theoretical notation

For a Feynman integral with standard propagators (of the $1/(m^2 - k^2 - i0)^{a_l}$ form) corresponding to a connected graph Γ, the alpha representation has the following form:

$$F_\Gamma(q_1, \ldots, q_n; d; a_1 \ldots, a_L) = \frac{i^{a+h(1-d/2)} \pi^{hd/2}}{\prod_l \Gamma(a_l)} \times \int_0^\infty \ldots \int_0^\infty \prod_l \alpha_l^{a_l - 1} \mathcal{U}_\Gamma^{-d/2} e^{i\mathcal{V}_\Gamma/\mathcal{U}_\Gamma - i\sum m_l^2 \alpha_l} d\alpha_1 \ldots d\alpha_L \, , \quad (2.1)$$

where L and h is, respectively, the number of lines (edges) and loops (independent circuits) of the graph, $n+1$ is the number of external vertices, $a = \sum a_l$, and

$$\mathcal{U}_\Gamma = \sum_{T \in T^1} \prod_{l \notin T} \alpha_l \, , \quad (2.2)$$

$$\mathcal{V}_\Gamma = \sum_{T \in T^2} \prod_{l \notin T} \alpha_l \left(q^T\right)^2 \, . \quad (2.3)$$

In (2.2), the sum runs over trees of the given graph, and, in (2.3), over *2-trees*, i.e. maximal subgraphs that do not involve loops and consist of two connectivity components; q^T is the sum of the external momenta that flow into one of the connectivity components of the 2-tree T. (It does not matter which component is taken because of the conservation law for the external momenta.) The products of the alpha parameters involved are taken over the lines that do not belong to the given (2-)tree T. The functions $\mathcal{U}$ and $\mathcal{V}$ are homogeneous functions of the alpha parameters with the homogeneity degrees h and $h+1$, respectively.

[1] Let us stress that Hepp and Speer sectors generally do not provide a resolution of the singularities in the parameter of dimensional regularization if some sum $\sum q_i$ is light-like.

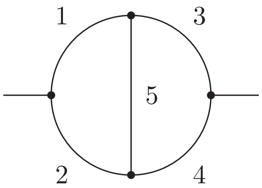

Figure 1. A two-loop propagator graph.

An overall integration can be performed to obtain another well-known parametric representation:[2]

$$F_\Gamma(q_1,\ldots,q_n;d;a_1\ldots,a_L) = \frac{\left(i\pi^{d/2}\right)^h \Gamma(a-hd/2)}{\prod_l \Gamma(a_l)}$$
$$\times \int_0^\infty \ldots \int_0^\infty \prod_l \alpha_l^{a_l-1} \delta\left(\sum \alpha_l - 1\right) \frac{\mathcal{U}_\Gamma^{a-(h+1)d/2}}{\left(-\mathcal{V}_\Gamma + \mathcal{U}_\Gamma \sum m_l^2 \alpha_l\right)^{a-hd/2}} \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_L \ . \quad (2.4)$$

According to the well-known folklore Cheng-Wu theorem one can choose any sum of the alpha variables in the argument of the delta function.

We imply that the graph Γ is a *connected* graph, i.e. any two vertices of Γ can be connected by a path in Γ. However, we are going to consider various subgraphs of the graph and they can be disconnected, i.e. consist of several connectivity components. A subgraph γ of Γ is determined by a subset of lines $\mathcal{L}(\gamma)$ and includes all the vertices incident to these lines. (Sometimes isolated vertices are added to a subgraph. For example, `Mathematica` produces isolated vertices as bi-connected components.) The number of loops of a subgraph equals to

$$h(\gamma) = L(\gamma) - V(\gamma) + c(\gamma) \ ,$$

where $V(\gamma)$ and $c(\gamma)$ are, respectively the numbers of the vertices and connectivity components.

An articulation vertex of a graph Γ is a vertex whose deletion disconnects Γ. Any graph with no articulation vertices is said to be *bi-connected* (or, *one-vertex-irreducible* (1VI)). Otherwise, it is called *one-vertex-reducible* (1VR). In other words, in a 1VR graph, one can distinguish two subsets of its lines and a vertex (an articulation vertex) such that any path between vertices from these two subsets goes through this vertex. ¿From now on let us suppose that we are dealing with a 1VI graph. It is natural to treat a single line as a 1VI graph since we cannot decompose it into two parts.

Any subgraph can be represented as the union of its 1VI components, i.e. maximal 1VI subgraphs. Consider, for example, the two-loop self-energy graph of figure 1. The subgraphs $\{1,2,5\}$ and $\{1,2,3,4\}$ are 1VI. The subgraph $\{1,2,3,5\}$ is 1VR and its 1VI components are $\{1,2,5\}$ and $\{3\}$. The subgraph $\{1,2,3\}$ is 1VR and its 1VI components are $\{1\}$, $\{2\}$ and $\{3\}$.

[2] So, the code of [19] works at least if each monomial of the α-variables in the function $-\mathcal{V}_\Gamma + \mathcal{U}_\Gamma \sum m_l^2 \alpha_l$ enters with a negative coefficient. This condition is a little bit relaxed within the code of [20] where combinations of the type $(\alpha_i - \alpha_j)^2$ are also admissible.

A set f of 1VI subgraphs is called an *ultraviolet (UV) forest* if the following conditions hold:

(i) for any pair $\gamma, \gamma' \in f$, we have either $\gamma \subset \gamma'$, $\gamma' \subset \gamma$ or $\mathcal{L}(\gamma \cap \gamma') = \emptyset$;

(ii) if $\gamma^1, \ldots, \gamma^n \in f$ and $\mathcal{L}(\gamma^i \cap \gamma^j) = \emptyset$ for any pair from this family, the subgraph $\cup_i \gamma^i$ is 1VR.

In other words, the number of loops in $\cup_i \gamma^i$ (where γ^i are disjoint with respect to lines and belong to a UV forest) is equal to the sum of the numbers of loops of γ^i. The term "UV" is used because the UV divergences are due to the integration over small values of α_l where the exponent in (2.1) is irrelevant and they are generated by the singularities of the factor $\mathcal{U}_\Gamma^{-d/2}$. We are going to show that the resolution of the UV singularities can be performed by the use of sectors associated with 1VI subgraphs.

For example, the set $\{1\}$, $\{2\}$, $\{3\}$ of subgraphs of figure 1 is a UV forest and $\{1,2,5\}$, $\{3\}$ is also a UV forest but the set $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ is not a UV forest because the condition (ii) breaks down.

Let $\mathcal{F}$ be a *maximal* UV forest (i.e. there are no UV forests that include $\mathcal{F}$) of a given graph Γ. An element $\gamma \in \mathcal{F}$ is called *trivial* if it consists of a single line and is not a loop line. Any maximal UV forest has h non-trivial and $L-h$ trivial elements.

Let us define the mapping $\sigma: \mathcal{F} \to \mathcal{L}$ such that $\sigma(\gamma) \in \mathcal{L}(\gamma)$ and $\sigma(\gamma) \notin \mathcal{L}(\gamma')$ for any $\gamma' \subset \gamma$, $\gamma' \in \mathcal{F}$. The inverse mapping $\sigma^{-1}: \mathcal{L} \to \mathcal{F}$ exists and can be defined as follows: $\sigma^{-1}(l)$ is the minimal element of the UV forest $\mathcal{F}$ that contains the line l. Let us denote by γ_+ the minimal element of $\mathcal{F}$ that strictly includes the given element γ.

For a given maximal UV forest $\mathcal{F}$, let us define the corresponding sector (f-sector) as

$$\mathcal{D}_\mathcal{F} = \left\{ \underline{\alpha} | \alpha_l \leq \alpha_{\sigma(\gamma)},\ l \in \gamma \in \mathcal{F} \right\} . \tag{2.5}$$

The intersection of two different f-sectors is of measure zero; the union of all the sectors gives the whole integration domain of the alpha parameters. For a given f-sector, let us introduce new variables labelled by the elements of $\mathcal{F}$,

$$\alpha_l = \prod_{\gamma \in \mathcal{F}:\ l \in \gamma} t_\gamma , \tag{2.6}$$

where the corresponding Jacobian is $\prod_\gamma t_\gamma^{L(\gamma)-1}$. The inverse formula is

$$t_\gamma = \begin{cases} \alpha_{\sigma(\gamma)}/\alpha_{\sigma(\gamma_+)} & \text{if } \gamma \text{ is not maximal} \\ \alpha_{\sigma(\gamma)} & \text{if } \gamma \text{ is maximal} \end{cases} . \tag{2.7}$$

Consider, for example, the following maximal UV forest $\mathcal{F}$ of figure 1 consisting of $\gamma^1 = \{1\}$, $\gamma^2 = \{2\}$, $\gamma^3 = \{3\}$, $\gamma^4 = \{1,2,5\}$, $\gamma^5 = \Gamma$. The mapping σ is $\sigma(\gamma^1) = 1$, $\sigma(\gamma^2) = 2$, $\sigma(\gamma^3) = 3$, $\sigma(\gamma^4) = 5$, $\sigma(\gamma^5) = 4$. The sector associated with this maximal UV forest is given by $\mathcal{D}_\mathcal{F} = \{\alpha_{1,2} \leq \alpha_5 \leq \alpha_4,\ \alpha_3 \leq \alpha_4\}$ and the sector variables are $t_{\gamma^1} = \alpha_1/\alpha_5$, $t_{\gamma^2} = \alpha_2/\alpha_5$, $t_{\gamma^3} = \alpha_3/\alpha_4$, $t_{\gamma^4} = \alpha_5/\alpha_4$, $t_{\gamma^5} = \alpha_4$.

All the maximal UV forests of the given graph can be constructed at least in two ways.

Way 1. We imply that the lines are enumerated. Let us consider the sequence of subgraphs γ_l consisting of lines $\{1, 2, \ldots, l\}$, respectively, with $l = 1, \ldots, L$. For each l, let us take the 1VI component of γ_l that includes the line l. The set of all these components is a maximal UV forest. Then we construct in a similar way the UV forests for other $L! - 1$ enumerations of the set of lines. After this we leave only distinct maximal UV forests.

Way 2. Since we consider a 1VI graph we include it into any maximal forest. Let us delete a line from it. The resulting graph is decomposed as the union of its 1VI components which we include into the maximal UV forest. Then we continue this process by deleting a line from some 1VI component which is not a single line, etc.

In the sector corresponding to a given maximal UV forest f, the function $\mathcal{U}_\Gamma$ takes the form

$$\mathcal{U}_\Gamma = \prod_{\gamma \in f} t_\gamma^{h(\gamma)} \left[1 + P_f\right] \,, \tag{2.8}$$

where P_f is a non-negative polynomial and the product is over elements of the given maximal UV forest f. We will call such a factorization *proper*.

This factorization formula is proved by constructing an appropriate tree. One uses the relation

$$\prod_{l \notin T} \alpha_l = \prod_{\gamma \in f} t_\gamma^{h(\gamma) + c(\gamma \cap T) - c(\gamma)} \,, \tag{2.9}$$

where T is a tree or a 2-tree so that the factorization reduces to constructing a tree that provides the minimal value of the non-negative quantity $c(\gamma \cap T) - c(\gamma)$. Let T_0 be the tree composed of all trivial elements of the given maximal UV-forest F. In other words, this tree can be constructed as follows. One uses an order of lines which was used within Way 1 for the construction of the given maximal UV forest f and includes the given line in the tree if a loop is not generated. One can observe that this tree T_0 provides the zero value of $c(\gamma \cap T_0) - c(\gamma)$ for all the elements of the given maximal forest.

To analyze convergence of the integral (2.1) large values of α_l (in particular, to reveal infrared (IR) divergences) one has to take into account the exponent as well. A possible way is to separate the integration over every α_l into $(0, 1)$ and $(1, \infty)$ and then to deal with each of these 2^L regions separately — see, e.g. [10, 11]. This can be enough for a general analysis but cannot be good from the practical point of view because the number of the resulting sectors will be too large. A more reasonable approach is to turn [5] to an integral with a compact integration domain, where both UV and IR divergences are somehow mixed up and manifest themselves as divergences at small values of parameters of integration. We will do this in the next section.

3 Strategies to reproduce Hepp and Speer sectors

In [14], the starting point is the alpha representation (2.4) where *primary sectors* are introduced. The set of primary sectors corresponds to the different choices of a line in the given graph. At this step, one chooses a line $l = 1, \ldots, L$ and defines a sector Δ_l by $\alpha_i \leq \alpha_l, \ i \neq l$ and the sector variables by $\alpha_i = t_i \alpha_l, \ i \neq l$. The integration over α_l is then taken due to the delta function whose argument is supposed to be the sum of all the variables minus one.

One can also start directly from (2.1) and introduce primary sectors $\alpha_i \leq \alpha_l,\ i \neq l$ there with the new variables, $\alpha_i = x_l\,\alpha_l$ belonging to a unit hypercube. For example, in the case of $l = L$, using the homogeneity properties of the functions in the representation and explicitly integrating over α_L we obtain the contribution of Δ_L as

$$F^{(L)} = \frac{\left(i\pi^{d/2}\right)^h \Gamma(a - hd/2)}{\prod_l \Gamma(a_l)} \int_0^1 \cdots \int_0^1 \prod_l^{L-1} x_l^{a_l-1}$$
$$\times \frac{\hat{\mathcal{U}}_\Gamma^{a-(h+1)d/2}}{\left[-\hat{\mathcal{V}}_\Gamma + \hat{\mathcal{U}}_\Gamma \left(\sum_{l=1}^{L-1} m_l^2 \prod_{l=l'}^{L-1} x_{l'} + m_L^2\right)\right]^{a-hd/2}} \mathrm{d}x_1 \ldots \mathrm{d}x_{L-1}\,, \quad (3.1)$$

where

$$\hat{\mathcal{U}}_\Gamma = \mathcal{U}(x_1, \ldots, x_{L-1}, 1)\,, \quad \hat{\mathcal{V}}_\Gamma = \mathcal{V}_\Gamma(x_1, \ldots, x_{L-1}, 1)\,. \quad (3.2)$$

Without loss of generality let us consider only this primary sector.

Let us remind that, for the case of non-zero masses, a general analysis of the factorization is not known even for Euclidean external momenta. Let us therefore turn to the pure massless case, as in [5]. Let us describe how sectors of Speer type can be introduced in such a way that the whole integrand of (3.1) has a proper factorization. As we could see in the previous section, the use of f-sectors provides a proper factorization (2.8) of the function $\mathcal{U}$ so that the factor $\hat{\mathcal{U}}_\Gamma^{a-(h+1)d/2}$ in (3.1) is properly factorized. However, these sectors generally do not provide a factorization of the second non-trivial factor. This can be seen using our example of figure 1. We are going to use smaller sectors which are in fact obtained from the f-sectors generated by the graph Γ by a further decomposition.

Let Γ^∞ be the graph obtained from Γ by adding a new vertex v^∞ and connecting it with all the external $n+1$ vertices by additional lines. These lines are only auxiliary and no propagators correspond to them. When writing down the function $\mathcal{U}$ for Γ^∞, let us include, by definition, these additional lines into any tree. Then in the case of two external vertices (i.e. for $n = 1$) we have

$$\mathcal{V}_\Gamma = \mathcal{U}_{\Gamma^\infty} q^2$$

where q is the only external momentum.

Let us define sectors in a way similar to the previous section but using, instead of 1VI subgraphs, another set of subgraphs which we call *s-irreducible*. If a subgraph γ does not have all the external vertices in the same connectivity component and if it is 1VI let us call it s-irreducible as well. If a subgraph γ has all the external vertices in the same connectivity component let us call it s-irreducible if the graph γ^∞ is 1VI. We will call an s-irreducible subgraph trivial if it is a single line which is not a loop line and which does not connect the external vertices.

The maximal forests consisting of s-irreducible subgraphs can be constructed again by Way 1 or Way 2.

We define sectors (we name them *Speer sectors*) in a way similar to the sectors discussed in the previous section. We introduce sector variables by the same formula (2.6) as above. The factorization of the function $\mathcal{V}$ follows from its definition (2.3) and the auxiliary relation (2.9). The 2-tree that provides a minimal value of the non-negative quantity $c(\gamma \cap T) - c(\gamma)$ can be constructed by a procedure similar to the procedure used for the function U: one considers the lines in the order used for the construction of the given f-forest by Way 1 and includes the given line into the 2-tree if a loop is not generated *and* if this is not the line whose inclusion would connect all the external vertices.

By construction, for such a 2-tree T_0, we obtain $c(\gamma \cap T_0) - c(\gamma) = \theta(\gamma)$ where $\theta(\gamma) = 1$ if the external vertices are connected in γ and $\theta(\gamma) = 0$ otherwise. Hence we obtain a proper factorization

$$\mathcal{V}_\Gamma = \prod_{\gamma \in f} t_\gamma^{h(\gamma)+\theta(\gamma)} \left[q_{T_0}^2 + P_\mathcal{V} \right] \ , \tag{3.3}$$

where $P_\mathcal{V}$ is a non-negative polynomial.

Obviously, the Speer sectors can be obtained from those associated with the graph Γ by a further decomposition, so that the factorization of the function $\mathcal{U}_\Gamma$ in the corresponding variables also holds and has the form similar to (2.8) with the same exponents.

Let us turn to the modern strategies of sector decompositions. After introducing primary sectors, one obtains the contribution (3.1) and other $L-1$ contributions of the same type. At each step, one chooses a subset of the indices $\nu = \{i_1, \ldots, i_k\}$ and an index i_r from this subset and defines a sector $x_i \leq x_{i_r}$, $i \neq i_r$ and the sector variables by $x_i = x'_i x_{i_r}$, $i \neq i_r$ To formulate a strategy of introducing iterative sectors one needs to fix rules for determining subsets ν at every recursive step.[3]

The first known sector decomposition strategy is described in [14]; three strategies guaranteed to terminate (A, B and C) and one strategy not guaranteed to terminate (X) are presented in [19]; they all are based on analyzing the functions $\mathcal{U}$ and $\mathcal{V}$ and choosing a subset of indices depending on their properties. Strategy S [20] is a bit different and is based on analyzing the polytopes of weights and their lowest faces. We present its definition in appendix.

We can now redefine the Hepp and Speer sectors iteratively. With the Hepp sectors, the situation is obvious: they are reproduced when we consider *maximal* subsets of lines at each step, i.e. with one line less than before this.

To reproduce the choice of Speer sectors within a sector decomposition let us remind the Way 2 to construct sectors. One has just to consider only subsets of the indices $\nu = \{i_1, \ldots, i_k\}$ that correspond to s-irreducible subgraphs.

We compared the number of sectors in numerous examples and discovered that the set of the sectors within Strategy S and the set of the Speer sectors was always the same. In particular, this was observed in the two examples of massless vertex diagrams shown in figure 2 at Euclidean external momenta, and in rather non-trivial examples of three four-loop propagator diagrams of figure 3 which are the most complicated master integrals among all four-loop massless propagator integrals.[4] The results of this analysis are shown in table 1 where the number of the sectors is shown. The first column stands for Strategy S and Speer

[3] Some strategies use more general sectors by comparing the integrations variables in different powers.
[4] Analytical results in expansion in ε up to ε^0 for these diagrams will be published soon [22].

JHEP05(2009)004

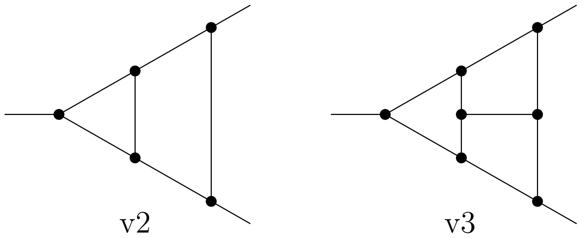

Figure 2. Vertex off-shell diagrams.

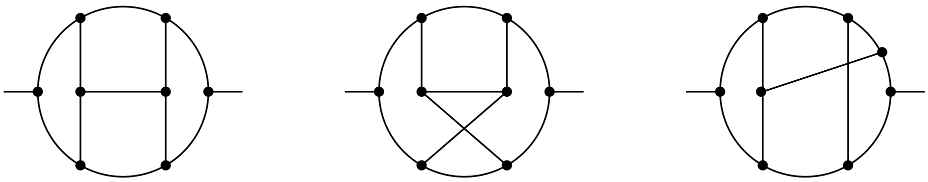

Figure 3. Most complicated four-loop propagator diagrams.

diagram	S	X
v2	102	102
v3	2160	2251
m61	26208	32620
m62	26304	27540
m63	27336	failed

Table 1. Comparison of numbers of sectors for different strategies.

sectors; the resulting number of the sectors is the same. For comparison, we included the second column for Strategy X [19] which has been proved to be very effective in a number of complicated calculations. Here 'failed' means that a factorization by sector decomposition has not been achieved for a reasonable amount of time (at least not for one day.)

Motivated by these observations, we formulated and proved a statement that for Feynman integrals with Euclidean external momenta *Strategy S and Speer sectors lead to the same sector decomposition.*

A proof of this theorem is presented in the appendix. Let us stress that the theorem relates strategies which originated from absolutely independent mathematical constructions.

4 Conclusion

In fact, our motivation to recall Speer sectors was to suggest to use them within `FIESTA` [20] since they are optimal for Feynman integrals at all Euclidean external momenta. However it turned out that these sectors are reproduced within Strategy S. Therefore we can conclude that Strategy S has chances to be an optimal universal strategy which always terminates and provides a proper factorization.

Let us remind that Speer sectors have a certain physical meaning: the integration over the sector variable t_γ is responsible for UV divergences and, if γ contains all the external vertices, for off-shell IR divergences. If on-shell and collinear divergences are present, Speer sectors are no longer applicable. Then one can use sector decompositions, e.g. within Strategy S, and this strategy could help to reveal the physical meaning of the sectors. To do this one might start with analyzing simple typical diagrams with on-shell or/and collinear divergences.

Acknowledgments

We are grateful to G. Heinrich for helpful discussions. Many thanks to N. Glover for kind hospitality during our visit to the Institute of Particle Physics Phenomenology in Durham, where a part of this work was done. The work was supported by the Russian Foundation for Basic Research through grant 08-02-01451.

A A proof of the equivalence of Strategy S and Speer sectors

Let us first remind the definition of Strategy S [20]. We consider the set of weights W of the polynomial $\mathcal{V}_\Gamma$ defined as the set of all possible $(a_1, \ldots, a_n)$ where $cx_1^{a_1} \ldots x_n^{a_n}$ is one of the monomials of $\mathcal{V}_\Gamma$. We will say that a weight is higher than another one if their difference is a set of non-negative numbers. If $\mathcal{V}_\Gamma$ has a unique lowest weight, a monomial can be factorized out, so no sector decomposition is needed. Hence it becomes reasonable to try to minimize the number of lowest weights of $\mathcal{V}_\Gamma$. We consider the convex hull of the lowest weights of W and choose one of its facets[5] G visible from the origin. Now let us take the normal vector v to G, consider the set $I = \{i | v_i \neq 0\}$ and separate the integration region in (3.1) into m parts by

$$S_l = \{(x_1, \ldots, x_n) | x_{i_l}^{a_{i_l}} \geq x_{i_k}^{a_{i_k}} \forall i_k \in I\},$$

where $\{i_1, \ldots i_m\} = I$, $n = L - 1$, and the exponents a_i are defined by

$$\begin{pmatrix} a_{i_1} \\ a_{i_2} \\ a_{i_3} \\ \vdots \\ a_{i_m} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix}^{-1} \begin{pmatrix} v_{i_1} \\ v_{i_2} \\ v_{i_3} \\ \vdots \\ v_{i_m} \end{pmatrix} \tag{A.1}$$

[5] A face of a convex polytope is its intersection with a hyperplane such that the polytope is contained in one of the corresponding half-spaces. A facet is a face of maximum dimension.

The variable replacement in S_l is defined by

$$\begin{aligned} x_i &= x'_i \forall i \notin I \\ x_{i_l} &= (x')_{i_l}^{v_{i_l}} \\ x_{i_k} &= (x')_{i_l}^{v_{i_k}} x'_{i_k} \ \forall i \in I, k \neq l. \end{aligned}$$

Let us turn back to Speer sectors. We need to introduce a few definitions. For a subgraph Γ' we denote the union of its nontrivial s-irreducible components with $C(\Gamma')$ and say that it is the clear version of Γ'. We say that a sequence $l_1, \ldots, l_j$ of lines of Γ' is admissible if the corresponding sequence of subgraphs $\Gamma = \Gamma, \ldots, \Gamma_{\{l_1,\ldots,l_j\}} = C(\Gamma \backslash \{l_1, \ldots, l_j\})$ satisfies the following condition: $\{l_{i+1}, \ldots, l_j\} \subset C(\Gamma_{\{l_1,\ldots,l_{i-1}\}} \backslash \{l_i\})$. In other words, it is possible to be removing those lines one by one according to Speer strategy rules.

Now let us prove an auxiliary statement.

1. The monomials in $\mathcal{V}_\Gamma$ are formed from variables corresponding to maximal admissible sequences of lines and they have the same homogeneity degree;
2. $\mathcal{V}_\Gamma = x_{l_1} \ldots x_{l_j} \mathcal{V}_{\Gamma_{\{l_1,\ldots,l_j\}}} + \tilde{\mathcal{V}}$, where $\tilde{\mathcal{V}}$ is some function not divisible by $x_{l_1} \ldots x_{l_j}$;
3. For any subgraph Γ', its line l and another line $l' \in \Gamma' \backslash C(\Gamma' \backslash \{l\})$ we have $C(\Gamma' \backslash \{l\}) = C(\Gamma' \backslash \{l'\})$;
4. Any monomial in $\tilde{\mathcal{V}}$ is divisible by some monomial in $\mathcal{V}_{\Gamma_{\{l_1,\ldots,l_j\}}}$;
5. For an s-irreducible $\Gamma' = C(\Gamma')$ the linear span of weights of $\mathcal{V}_{\Gamma'}$ coincides with the linear space of all possible weights of monomials with variables corresponding to Γ'.

Observe that if $\{l_1, \ldots, l_j\}$ is a maximal admissible sequence then the corresponding function $\mathcal{V}_{\Gamma_{\{l_1,\ldots,l_j\}}}$ is independent of the x-variables.

The first statement is more or less obvious and can be considered as a reformulation of the definition of $\mathcal{V}$ and the second is its direct consequence.

The third statement: the obvious part it that if a line is contained in $C(\Gamma' \backslash \{l\})$ then it belongs to some s-irreducible component not containing l' therefore $C(\Gamma' \backslash \{l\}) \supset C(\Gamma' \backslash \{l'\})$. Now it is enough to prove that $l \notin C(\Gamma' \backslash \{l'\})$. Suppose the contrary. Then we can construct an admissible sequence starting with l' and l, so $C(\Gamma' \backslash \{l\}) = C(\Gamma' \backslash \{l', l\})$. However, this is impossible because the lengths of all maximal admissible sequences coincide and depend only on the graph, so that we come to a contradiction.

The fourth statement can be proved by induction if we can prove the following: for a subgraph Γ' and a line $l \in \Gamma'$ we have $\mathcal{V}_{\Gamma'} = x_l \mathcal{V}_{C(\Gamma' \backslash \{l\})} + \tilde{\mathcal{V}}$. Suppose the contrary and pick a monomial $M = x_{l_1} \ldots x_{l_j}$ in $\mathcal{V}$ not divisible by any monomial in $\mathcal{V}_{C(\Gamma' \backslash \{l\})}$. Consider the maximal admissible sequence of lines $\{l_1, \ldots, l_k\}$ corresponding to the variables of M such that $x \in C(\Gamma' \backslash \{l_1, \ldots, l_k\})$. We can deduce that $k = j - 1$; indeed if k would be less, then there would be two lines corresponding to variables in M such that the deletion of any of those and clearing the graph results in removal of x. However due to part 3 this means that the deletion of any of those lines and clearing leads to the removal of the other line, and that is impossible since $\{l_1,\ldots,l_j\}$ is admissible, so that we come to a contradiction.

An alternative proof of the fourth statement: it is necessary to find a 2-tree T_1 contributing to $\mathcal{V}_{\Gamma_{\{l_1,\ldots,l_j\}}}$ such that $T_2 \supset T_1$ for a given 2-tree T_2 contributing to $\tilde{\mathcal{V}}$. To do this, we first delete all the lines $l_1,\ldots,l_j$ from the given 2-tree T_2. Then we start from the rest of the lines of T_2 and construct a 2-tree of $\Gamma_{\{l_1,\ldots,l_j\}}$ using Way 1 of section 2.

Let us prove the fifth statement by induction on the number of lines in a graph.

Suppose the contrary. Then there is some non-zero vector v orthogonal to all the weights of $\mathcal{V}_{\Gamma'}$. We have $\sum v_i a_i = 0$ for all monomials $x_1^{a_1}\ldots x_n^{a_n}$ of $\mathcal{V}_{\Gamma'}$. Let us choose some line l of Γ' and the corresponding variable v. The $\sum v_i a_i$ is constant for all monomials in $\mathcal{V}_{C(\Gamma'\backslash\{l\})}$. But the function $\mathcal{V}_{C(\Gamma'\backslash\{l\})}$ factorizes into the product of similar functions for s-irreducible components, therefore $\sum v_i a_i$ is constant also for all s-irreducible components of $C(\Gamma'\backslash\{l\})$. By induction we know that the indices v_i are constant on each of those irreducible components (the induction statement is equivalent to the statement that only the vectors proportional to $\{1,\ldots,1\}$ have the same scalar product with all weights). Therefore we have shown that if two lines can be included into one s-irreducible subgraph then the indices v_i on those lines coincide. Consequently all indices v_i coincide, so $\sum a_i = 0$ for all monomials. A contradiction.

Now we are ready to prove that the strategies result in the same sectors. To do this we are going to prove that the steps of different strategies coincide. We are going to prove it by induction, where the induction statement is that after i steps corresponding to removing $\{l_1,\ldots,l_k\}$ the sectors coincide and the minimal weights of the current function are exactly the weights of $\mathcal{V}_{C(\Gamma')}$ where $\Gamma' = C(\Gamma\backslash\{l_1,\ldots,l_k\})$.

Let us prove the induction step. First of all let us suppose that the current graph Γ' is s-irreducible. The Speer sector strategy suggests to compare all variables corresponding to lines of Γ' and to choose the highest one. However due to the induction statement, the lowest weights of the current function for strategy S are the weights of $\mathcal{V}_{C(\Gamma')}$. Those monomials are of the same degree but according to statement 5, the linear span of those weights coincides with all possible weights of monomials with variables corresponding to Γ'. Consequently the only way to choose a face of maximal rank visible from the origin is to make it coinciding with the set of weights of $\mathcal{V}_{C(\Gamma')}$. The normal vector of this face has coinciding coordinates, therefore the sector decomposition steps coincide.

Now let us analyze what happens with the function $\mathcal{V}_{\Gamma'}$ after such a sector decomposition step. Let us choose a sector and a variable x greater than the others. According to statement 2, $\mathcal{V}_{\Gamma'} = x\mathcal{V}_{\Gamma'_{\{l\}}} + \tilde{\mathcal{V}}$ where $\tilde{\mathcal{V}}$ is not divisible by x. The sector decomposition step consists of multiplying all other variables by x, hence after the step it is the function $\mathcal{V}_{\Gamma'_{\{l\}}}$ that contains the monomial of minimal degree after the variable replacement. Moreover, according to statement 4, all monomials of $\tilde{\mathcal{V}}$ are divisible by some monomial of $\mathcal{V}_{\Gamma'_{\{l\}}}$, and it is exactly what we need to finish the induction step.

The only thing left to do is to analyze what happens for s-reducible subgraphs. Inside the Speer sectors strategy we have to treat all irreducible components independently. However the function $\mathcal{V}_{\Gamma'}$ factorizes into functions corresponding to those components, therefore the same is valid for Strategy S.

JHEP05(2009)004

References

[1] N.N. Bogoliubov and D.V. Shirkov, *Introduction to theory of quantized fields*, third edition, Wiley, New York U.S.A. (1983).

[2] N. Nakanishi, *Graph theory and Feynman integrals*, Gordon and Breach, New York U.S.A. (1971).

[3] K. Hepp, *Proof of the Bogolyubov-Parasiuk theorem on renormalization*, *Commun. Math. Phys.* **2** (1966) 301 [SPIRES].

[4] E.R. Speer, *Analytic renormalization*, *J. Math. Phys.* **9** (1969) 1404.

[5] E.R. Speer, *Ultraviolet and infrared singularity structure of generic Feynman amplitudes*, *Annales Poincare Phys. Theor.* **23** (1975) 1 [SPIRES].

[6] M.C. Bergere and J.B. Zuber, *Renormalization of Feynman amplitudes and parametric integral representation*, *Commun. Math. Phys.* **35** (1974) 113 [SPIRES];
M.C. Bergere and Y.-M.P. Lam, *Bogolyubov-Parasiuk theorem in the alpha parametric representation*, *J. Math. Phys.* **17** (1976) 1546 [SPIRES].

[7] M.C. Bergere, C. de Calan and A.P.C. Malbouisson, *A theorem on asymptotic expansion of Feynman amplitudes*, *Commun. Math. Phys.* **62** (1978) 137 [SPIRES].

[8] P. Breitenlohner and D. Maison, *Dimensional renormalization and the principle*, *Commun. Math. Phys.* **52** (1977) 11 [SPIRES]; *Dimensionally renormalized Green's functions for theories with massless particles. 1*, *Commun. Math. Phys.* **52** (1977) 39 [SPIRES]; *Dimensionally renormalized Green's functions for theories with massless particles. 2*, *Commun. Math. Phys.* **52** (1977) 55 [SPIRES].

[9] K. Pohlmeyer, *Large momentum behavior of the Feynman amplitudes in the* π^4 *in four-dimensions theory*, *J. Math. Phys.* **23** (1982) 2511 [SPIRES].

[10] O.I. Zavialov, *Renormalized quantum field theory*, Kluwer Academic Publishers, Dodrecht The Netherlands (1990);
V.A. Smirnov, *Asymptotic expansions in limits of large momenta and masses*, *Commun. Math. Phys.* **134** (1990) 109 [SPIRES].

[11] V.A. Smirnov, *Applied asymptotic expansions in momenta and masses*, *Springer Tracts Mod. Phys.* **177** (2002) 1 [SPIRES].

[12] G. 't Hooft and M.J.G. Veltman, *Regularization and renormalization of gauge fields*, *Nucl. Phys.* **B 44** (1972) 189 [SPIRES].

[13] C.G. Bollini and J.J. Giambiagi, *Dimensional renormalization: the number of dimensions as a regularizing parameter*, *Nuovo Cim.* **B 12** (1972) 20 [SPIRES].

[14] T. Binoth and G. Heinrich, *An automatized algorithm to compute infrared divergent multi-loop integrals*, *Nucl. Phys.* **B 585** (2000) 741 [hep-ph/0004013] [SPIRES]; *Numerical evaluation of multi-loop integrals by sector decomposition*, *Nucl. Phys.* **B 680** (2004) 375 [hep-ph/0305234] [SPIRES]; *Numerical evaluation of phase space integrals by sector decomposition*, *Nucl. Phys.* **B 693** (2004) 134 [hep-ph/0402265] [SPIRES].

[15] G. Heinrich, *Sector decomposition*, *Int. J. Mod. Phys.* **A 23** (2008) 1457 [arXiv:0803.4177] [SPIRES].

[16] V.A. Smirnov, *Analytical result for dimensionally regularized massless on-shell planar triple box*, *Phys. Lett.* **B 567** (2003) 193 [hep-ph/0305142] [SPIRES].

JHEP05(2009)004

[17] T. Gehrmann, G. Heinrich, T. Huber and C. Studerus, *Master integrals for massless three-loop form factors: one-loop and two-loop insertions*, Phys. Lett. **B 640** (2006) 252 [hep-ph/0607185] [SPIRES];
G. Heinrich, T. Huber and D. Maître, *Master integrals for fermionic contributions to massless three-loop form factors*, Phys. Lett. **B 662** (2008) 344 [arXiv:0711.3590] [SPIRES].

[18] R. Boughezal and M. Czakon, *Single scale tadpoles and $\mathcal{O}(G(F)m(t)^2\alpha_s^3)$ corrections to the rho parameter*, Nucl. Phys. **B 755** (2006) 221 [hep-ph/0606232] [SPIRES].

[19] C. Bogner and S. Weinzierl, *Resolution of singularities for multi-loop integrals*, Comput. Phys. Commun. **178** (2008) 596 [arXiv:0709.4092] [SPIRES]; *Blowing up Feynman integrals*, Nucl. Phys. Proc. Suppl. **183** (2008) 256 [arXiv:0806.4307] [SPIRES].

[20] A.V. Smirnov and M.N. Tentyukov, *Feynman Integral Evaluation by a Sector decomposiTion Approach (FIESTA)*, Comput. Phys. Commun. **180** (2009) 735 [arXiv:0807.4129] [SPIRES].

[21] V.N. Velizhanin, *Leading transcedentality contributions to the four-loop universal anomalous dimension in $N = 4$ SYM*, arXiv:0811.0607 [SPIRES];
G. Bell, *NNLO corrections to inclusive semileptonic B decays in the shape-function region*, Nucl. Phys. **B 812** (2009) 264 [arXiv:0810.5695] [SPIRES];
Y. Kiyo, D. Seidel and M. Steinhauser, *$\mathcal{O}(\alpha\alpha_s)$ corrections to the $\gamma t\bar{t}$ vertex at the top quark threshold*, JHEP **01** (2009) 038 [arXiv:0810.1597] [SPIRES];
R. Bonciani and A. Ferroglia, *Two-Loop QCD corrections to the heavy-to-light quark decay*, JHEP **11** (2008) 065 [arXiv:0809.4687] [SPIRES];
A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *Fermionic contributions to the three-loop static potential*, Phys. Lett. **B 668** (2008) 293 [arXiv:0809.1927] [SPIRES].

[22] P.A. Baikov and K.G. Chetyrkin, *Four loop massless propagators: an algebraic evaluation of all master integrals* in preparation.

JHEP05(2009)004